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Propagation Property for Nonlinear Parabolic Equations of p-Laplacian-Type[1]

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Abstract. We study propagation property for one-dimensional nonlinear diffusion equations with convection-absorption, including the prototype model

$$\partial_t(u^m) - \partial_x(|\partial_x u|^{p-1}\partial_x u) - \mu|\partial_x u|^{q-1}\partial_x u + \lambda u^k = 0,$$

where $m, p, q, k > 0$, and n-dimensional simplified variant

$$\partial_t(u^m) - \Delta_{p+1} u = 0,$$

where $\Delta_{p+1} u = \operatorname{div}(|\nabla u|^{p-1}\nabla u)$. Among the conclusions, we make complete classification of the parameters in the first equation to distinguish its propagation property. For the second equation we rigorously prove that perturbation of the nonnegative solutions propagates at finite speed if and only if $m < p$.

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Keywords: propagation, diffusion, convection, super- and sub-solutions, parabolic p-Laplacian equation

1. **Introduction**

Consider the following nonlinear parabolic equation

$$\partial_t\varphi(u) - \partial_x(|\partial_x u|^{p-1}\partial_x u) - A(|\partial_x u|)\partial_x u + B(u) = 0, \tag{1.1}$$

where $p > 0$, functions $\varphi(z)$, $A(z)$, and $B(z)$ are continuous. In addition, $\varphi(z)$ is increasing, and $B(z)$ is nonnegative.

This equation models diffusion-convection-absorption process in one dimension. Degenerate-singular diffusion occurs when $p \neq 1$, $\varphi'(z) = 0$ or $\varphi'(z) = +\infty$ at some $z \geq 0$, while singular convection appears if $A(|z|) \to \infty$ as $z \to 0$.

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Moreover, equation (1.1) is also related with n-dimensional nonlinear diffusion equation

$$\partial_t \varphi(u) - \Delta_{p+1} u = 0 \quad \text{in} \quad \mathbb{R}^n \times (0, \infty), \tag{1.2}$$

where $\Delta_{p+1} u := \operatorname{div}(|\nabla u|^{p-1} \nabla u)$. In fact, if a solution to (1.2) is of radially symmetric form, i.e., $u(x,t) = u(r,t)$, where $r = |x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$, then the equation becomes

$$\partial_t \varphi(u) - \partial_r(|\partial_r u|^{p-1} \partial_r u) - (n-1) r^{-1} |\partial_r u|^{p-1} \partial_r u = 0, \tag{1.3}$$

of the form like equation (1.1).

In this work we are interested in propagation properties, such as finite- or infinite-speed propagation of perturbation for the nonnegative solutions. There are many authors who studied propagation property for linear and nonlinear parabolic equations in one-dimensional case [3, 5, 7, 9, 10, 12, 14, 15], and multi-dimensional case [2, 6, 11, 16, 17, 18, 19]. As we know, however, the convection term studied in most literature is unknown-dependent [5, 6, 11, 14, 15], not derivative-dependent like (1.1).

For the special equation of (1.2),

$$\partial_t (u^m) - \Delta_{p+1} u = 0 \quad \text{in} \quad \mathbb{R}^n \times (0, \infty), \tag{1.4}$$

one has long believed that the sufficient and necessary condition for finite speed propagation of perturbation of solutions is $m < p$. This has been proved in one-dimensional case (cf. [3]). However, we have never found its complete multi-dimensional proof in the literature, especially for the necessity of condition $m < p$, even though the special case $p = 1$ was indeed investigated. In the present paper we prove general results for equation (1.2) which imply this sufficient and necessary condition as special conclusion in particular situation.

As for the existence of solutions to various initial-boundary value problems for nonlinear diffusion equations, with or without convection-absorption, one may see [1, 3, 7, 8, 13, 18] and references therein.

Throughout this paper we imposed the following conditions:

(1) $\varphi(z) \in C(\overline{\mathbb{R}}_+) \cap C^1(\mathbb{R}_+)$, $\varphi(0) = 0$, $\varphi'(z) > 0$ for $z > 0$;
(2) $A(z) \in C(\mathbb{R}_+)$;
(3) $B(z) \in C(\overline{\mathbb{R}}_+)$, $B(0) = 0$, $B(z) \geq 0$ for $z \geq 0$.

The rest of article is organized as follows: In Section 2 we introduce some notations and outline working device in the present paper. The main results are demonstrated in Section 3 and Section 4, respectively for one-dimensional equation (1.1) and for n-dimensional variant (1.2).

2. Some Definitions and Working Machinery

First of all we recall the notation of solutions.

Definition 2.1. A function $u(x,t)$ is called a super-(sub-) solution of equation (1.1) in Q, where $Q = (x_1, x_2) \times (0, T]$ with $0 \le x_1 < x_2 \le \infty$ and $0 < T < \infty$, if

(a) $u \in C(\overline{Q})$, $u \ge 0$ in Q, $\partial_x u \in C(Q)$, and
(b) in the sense of distribution

$$\partial_t \varphi(u) - \partial_x(|\partial_x u|^{p-1}\partial_x u) - A(|\partial_x u|)\partial_x u + B(u) \ge 0 \quad (\le 0).$$

$u(x,t)$ is said to be a solution if it is both a super- and sub-solution.

Next we explain some terms to describe propagation properties for solutions.

Definition 2.2. It is said that the equation (1.1) admits the finite speed propagation of perturbation (FSP for short) if for every solution $u(x,t)$ in $Q_T = (0,\infty) \times (0,T)$ with some $T > 0$, satisfying

$$u(x,0) = 0 \quad \text{for} \quad x \ge a$$

with some $a \ge 0$, there exists $\tau \in (0,T)$ and $r(t) \in (0,\infty)$, $t \in [0,\tau]$, such that

$$u(x,t) = 0 \quad \text{for} \quad x \ge r(t).$$

Definition 2.3. It is said that the equation (1.1) admits the infinite speed propagation of perturbation (ISP for short) if for every solution $u(x,t)$ in Q_T, satisfying $u(0,0) > 0$, there exists $\tau \in (0,T)$ such that

$$u(x,t) > 0 \quad \text{for all} \quad x > 0, \quad 0 < t < \tau.$$

Remark 2.4. Above-mentioned propagation properties are all concerned for x-forward direction. Obviously, for x-backward direction propagation properties could be described similarly.

In order to characterize propagation properties we make use of super- and sub-solution method. To this purpose comparison principle plays an important role.

Lemma 2.5 (Comparison Principle)**.** *Let $u_1(x,t)$ and $u_2(x,t)$ be super- and sub-solutions of (1.1) in Q, respectively, and $u_1 \ge u_2$ on $\overline{Q} \setminus Q$. Then $u_1 \ge u_2$ on the whole $\overline{Q}$. Here $Q = (x_1, x_2) \times (0,T]$ as above.*

With regard to the study of comparison principle see, e.g., [1, 4, 14].

Now we show two sufficient conditions for FSP and ISP properties, respectively. The argument also explains our main idea to construct super- and sub-solutions.

Lemma 2.6. *If there exist $\omega \in \mathbb{R}$, $\sigma > 0$, $f(z) \in C(\overline{\mathbb{R}}_+) \cap C^1(\mathbb{R}_+)$ satisfying $f(0) = 0$, $f'(z) > 0$ for $0 < z < \sigma$, and $\int_0^\sigma \frac{dz}{f(z)} < \infty$, such that*

$$\omega\varphi'(z) + A(f(z)) - ([f(z)]^p)' + B(z)/f(z) \ge 0 \tag{2.1}$$

in the interval $0 < z < \sigma$, then equation (1.1) admits FSP property.

Proof. Let $u(x,t)$ be a solution of (1.1) in $Q_T = (0,\infty)\times(0,T)$, and satisfy

$$u(x,0) = 0 \quad \text{for} \quad x \geq a \geq 0.$$

In order to show FSP property of u we construct a super-solution as follows.

Let $z(\zeta)$ solve the following equation:

$$\int_0^z \frac{ds}{f(s)} = \zeta_+ \quad \text{for} \quad \zeta \leq \zeta_0 := \int_0^\sigma \frac{ds}{f(s)}, \tag{2.2}$$

where $\zeta_+ = \max\{\zeta, 0\}$. Then $z'(\zeta) = f(z)$, and $0 \leq z(\zeta) \leq \sigma$ for $\zeta \leq \zeta_0$.

Without loss of generality, let $\omega > 0$ in (2.1). Set $x_0 \in (0,\zeta_0)$, and $t_0 = (\zeta_0 - x_0)/\omega$. Now consider function $v(x,t) = z(\omega t - x + x_0 + a)$ in $Q_{t_0} = (a,\infty)\times(0,t_0)$.

We have $0 \leq v(x,t) \leq z(\omega t_0 + x_0) = \sigma$. From (2.1) and (2.2) it is not hard to verify that $v(x,t)$ is a super-solution of (1.1) in Q_{t_0}. In fact,

$$\begin{aligned}\partial_t\varphi(v) - &\partial_x(|\partial_x v|^{p-1}\partial_x v) - A(|\partial_x v|)\partial_x v + B(v)\\ &= \varphi'(z)\omega f(z) - ([f(z)]^p)' f(z) + A(f(z))f(z) + B(z)\\ &= f(z)\{\varphi'(z)\omega - ([f(z)]^p)' + A(f(z)) + B(z)/f(z)\} \geq 0.\end{aligned}$$

Besides, we see that

$$u(x,0) = 0 \leq v(x,0) = z(-x + x_0 + a) \quad \text{for } x \geq a;$$

and by the continuity there exists $\tau \in (0, \min\{t_0, T\})$ so that

$$u(a,t) \leq z(\omega t + x_0) = v(a,t) \quad \text{for} \quad 0 \leq t \leq \tau.$$

According to the comparison principle we obtain that $u \leq v$ on $\overline{Q}_\tau = [a,\infty]\times[0,\tau]$. Particularly,

$$u(x,t) = 0 \quad \text{for} \quad x \geq \omega t + x_0 + a, \quad 0 \leq t \leq \tau,$$

i.e., FSP is admitted. □

Lemma 2.7. *Assume that $A(z)$ is non-increasing. If for every $\omega \in \mathbb{R}$ there exists $\sigma > 0$, $f \in C(\overline{\mathbb{R}}_+) \cap C^1(\mathbb{R}_+)$, satisfying $f(0) = 0$, $f'(z) > 0$ for $0 < z < \sigma$, and $\int_0^\sigma \frac{dz}{f(z)} = \infty$, such that*

$$\omega\varphi'(z) + A(f(z)) - ([f(z)]^p)' + B(z)/f(z) \leq 0 \quad \textit{for } 0 < z \leq \sigma, \tag{2.3}$$

then (1.1) admits ISP property.

Proof. Let $u(x,t)$ be a solution of (1.1) in Q_T with $u(0,0) > 0$. Then by the continuity there exists $\tau > 0$ so that

$$\inf_{0<t<\tau} u(0,t) = \eta > 0.$$

We claim that $u > 0$ in Q_τ.

Fix an arbitrary point $(x_0, t_0) \in Q_\tau$, $w = 2x_0/t_0$, and choose $\sigma \in (0,\eta]$ such that $\int_0^\sigma \frac{dz}{f(z)} = \infty$ and (2.3) is fulfilled. Denote

$$\tilde{f}(z) := ([f(z)]^p + \varepsilon^p z^{\beta p})^{\frac{1}{p}},$$

where $\beta \in (0,1)$ is fixed and $\varepsilon > 0$ will be chosen later. Obviously, $\tilde{f}(z) \geq f(z)$, and $\tilde{f}(z) \geq \varepsilon z^\beta$ for $z > 0$, and so $\int_0^\sigma \frac{dz}{\tilde{f}(z)} < \infty$. Moreover, from (2.3) it is easy to see that

$$\begin{aligned} (2.4) \qquad & \omega\varphi'(z) + A(\tilde{f}(z)) - ([\tilde{f}(z)]^p)' + B(z)/\tilde{f}(z) \\ & \leq \omega\varphi'(z) + A(f(z)) - ([f(z)]^p + \varepsilon^p z^{\beta p})' + B(z)/f(z) \\ & \leq \omega\varphi'(z) + A(f(z)) - ([f(z)]^p)' + B(z)/f(z) \leq 0, \quad 0 < z < \sigma. \end{aligned}$$

Since $\int_0^\sigma \frac{dz}{f(z)} = \infty$, we can choose $\varepsilon > 0$ so small that

$$\zeta_0 := \int_0^\sigma \frac{ds}{\tilde{f}(s)} \geq \omega\tau.$$

Define function $\tilde{z}(\zeta)$ by the following integral equality

$$(2.5) \qquad \int_0^{\tilde{z}} \frac{ds}{\tilde{f}(s)} = \zeta_+ \quad \text{for} \quad \zeta \leq \zeta_0.$$

Let $v(x,t) = \tilde{z}(\omega t - x)$. Then from (2.4), (2.5) it follows that in Q_τ function $v(x,t)$ is a sub-solution of (1.1) and $0 \leq v(x,t) \leq \tilde{z}(\omega\tau) \leq \sigma$. In addition,

$$v(x,0) = \tilde{z}(-x) = 0 \leq u(x,0) \quad \text{for} \quad x \geq 0,$$

$$v(0,t) \leq \sigma \leq u(0,t) \quad \text{for} \quad 0 \leq t \leq \tau.$$

Therefore, by the comparison principle, $v \leq u$ in Q_τ. In particular,

$$u(x_0,t_0) \geq v(x_0,t_0) = \tilde{z}(\omega t_0 - x_0) > 0,$$

since $\omega t_0 - x_0 = x_0 > 0$. From the arbitrariness of (x_0,t_0) in Q_τ the lemma follows. □

3. **One-Dimensional Equations**

We start with the equation with no absorption

$$(3.1) \qquad \partial_t\varphi(u) - \partial_x(|\partial_x u|^{p-1}\partial_x u) - A(|\partial_x u|)\partial_x u = 0.$$

Theorem 3.1. *(1) If for every $\omega \in \mathrm{R}$ there exists a $\sigma > 0$ so that*

$$\omega\varphi'(z) + A(z) \leq pz^{p-1}, \quad 0 < z \leq \sigma,$$

then equation (3.1) *admits ISP property.*

(2) Equation (3.1) *admits FSP property if for some $\sigma > 0$,*

$$A(z) \geq 0 \quad \textit{in } (0,\sigma), \quad \textit{and} \quad \int_0^\sigma \frac{dz}{[\varphi(z)]^{1/p}} < +\infty.$$

Proof. (1): Choose $f(z) = z$, i.e., $\int_0^\sigma \frac{dz}{f(z)} = \infty$. Then for any ω we have,

$$\omega\varphi'(z) + A(f(z)) - ([f(z)]^p)' = \omega\varphi'(z) + A(z) - (z^p)' \leq 0, \quad 0 < z \leq \sigma,$$

for some $\sigma > 0$. The conclusion then follows from Lemma 2.7.

(2): Let $f(z) = [\varphi(z)]^{1/p}$, then

$$([f(z)]^p)' = \varphi'(z) \le A(f(z)) + \varphi'(z)$$

holds at least for $z > 0$ small enough, and thus, by Lemma 2.6, FSP is admitted. □

Now we consider the equation with absorption but no convection

$$\partial_t\varphi(u) - \partial_x(|\partial_x u|^{p-1}\partial_x u) + B(u) = 0. \tag{3.2}$$

This part of results comes from [15], but more accurate and with some improvement.

Theorem 3.2. *(1) Equation* (3.2) *admits the finite speed propagation if there exist $a \ge 0$, $b \ge 0$, and $\sigma > 0$, so that*

$$\int_0^\sigma \frac{dz}{\{a[\varphi(z)]^{(p+1)/p} + b\int_0^z B(s)ds\}^{1/(p+1)}} < \infty. \tag{3.3}$$

In particular, FSP occurs if

$$\int_0^\sigma \frac{dz}{[\int_0^z B(s)ds]^{1/(p+1)}} < \infty \quad \text{or} \quad \int_0^\sigma \frac{dz}{[\varphi(z)]^{1/p}} < \infty.$$

(2) Equation (3.2) *admits the infinite speed propagation if for all $\omega \ge 0$, and $\sigma > 0$ (sufficiently small), we have*

$$\int_0^\sigma \frac{dz}{\{\omega\varphi(z) + [\int_0^z B(s)ds]^{p/(p+1)}\}^{1/p}} = \infty. \tag{3.4}$$

Note that in this case

$$\int_0^\sigma \frac{dz}{[\int_0^z B(s)ds]^{1/(p+1)}} = \infty \quad \text{and} \quad \int_0^\sigma \frac{dz}{[\varphi(z)]^{1/p}} = \infty.$$

Proof. (1): Let

$$f(z) = \left\{[\omega\varphi(z)]^{\frac{p+1}{p}} + \frac{p+1}{p}\int_0^z B(s)ds\right\}^{\frac{1}{p+1}}.$$

The condition (3.3) implies that $\int_0^\sigma dz/f(z) < \infty$ for some $\omega \ge 0$. Moreover, it is clear that $f(z) \ge [\omega\varphi(z)]^{1/p}$, and

$$\begin{aligned}([f(z)]^p)'f(z) &= \left(\frac{p}{p+1}[f(z)]^{p+1}\right)' \\ &= B(z) + \omega\varphi'(z)[\omega\varphi(z)]^{\frac{1}{p}} \le B(z) + \omega\varphi'(z)f(z)\end{aligned}$$

Hence the FSP follows from Lemma 2.6.

(2): Let

$$f(z) = \left\{\omega\varphi(z) + \left[\frac{p+1}{p}\int_0^z B(s)ds\right]^{\frac{p}{p+1}}\right\}^{\frac{1}{p}},$$

then for all $\omega \geq 0$, $\int_0^\sigma dz/f(z) = \infty$ follows from (3.4). We further have that $f(z) \geq [(1+1/p)\int_0^z B(s)ds]^{1/(p+1)}$, and

$$([f(z)]^p)' = \omega\varphi'(z) + \frac{B(z)}{[(1+1/p)\int_0^z B(s)ds]^{1/(p+1)}} \geq \omega\varphi'(z) + \frac{B(z)}{f(z)}.$$

This indicates, by Lemma 2.7, that the ISP is admitted. □

To conclude this section, we consider fully diffusion-convection-absorption equation of power-like nonlinearity:

$$\partial_t(u^m) - \partial_x(|\partial_x u|^{p-1}\partial_x u) - \mu|\partial_x u|^{q-1}\partial_x u + \lambda u^k = 0, \tag{3.5}$$

where $m, p, q, k > 0$, and $\mu \in \mathbb{R}$, $\lambda \geq 0$. To state the result more concisely we denote $q^* = H_\mu(q, m)$, $k^* = H_\lambda(k, m)$ with

$$H_\epsilon(h, m) := \begin{cases} h, & \epsilon > 0, \\ m, & \epsilon = 0. \end{cases}$$

Again applying preceding method to this equation yields

Theorem 3.3. *Consider two situations as follows:*

(1) *Let $\mu \geq 0$. If $\min\{m, q^*, k^*\} < p$, then FSP occurs; otherwise, (i.e., $\min\{m, q^*, k^*\} \geq p$) we have ISP for the equation.*
(2) *Let $\mu < 0$. If $\min\{m, k^*\} < \min\{p, q\}$, or $1 \geq m = q < p$, then FSP happens; otherwise, ISP must appear.*

Proof. Again we construct super- and sub-solutions of the form

$$v(x,t) = z(\omega t - x + x_0),$$

where $z = z(\zeta)$ satisfies, with some $\beta \in (0,1)$ to be determined,

$$z' = az^\beta, \quad z(\zeta) = 0 \ \text{ for } \zeta \leq 0.$$

Namely,

$$\int_0^z \frac{ds}{as^\beta} = \zeta_+.$$

Therefore, we calculate that

$$\begin{aligned} &\partial_t(v^m) - \partial_x(|\partial_x v|^{p-1}\partial_x v) - \mu|\partial_x v|^{q-1}\partial_x v + \lambda v^k \\ &= m\omega a z^{m-1+\beta} - a^{p+1}p\beta z^{p\beta-1+\beta} + \mu a^q z^{q\beta} + \lambda z^k. \end{aligned}$$

In the case (1) $\mu \geq 0$, if $\min\{m, q^*, k^*\} < p$, we may choose $\beta \in (0,1)$ close to 1 so that

$$m \leq p\beta \ \text{ or } \ q^*\beta \leq p\beta - 1 + \beta \ \text{ or } \ k^* \leq p\beta - 1 + \beta,$$

and hence, with some $\omega \geq 0$, $a > 0$,

$$m\omega a z^{m-1+\beta} - a^{p+1}p\beta z^{p\beta-1+\beta} + \mu a^q z^{q\beta} + \lambda z^k \geq 0 \tag{3.7}$$

holds for $z > 0$ sufficiently small. This indicates that $v(x,t)$ is a super-solution with compact support $\operatorname{supp} v(\cdot, t)$ for all $t \geq 0$ small enough, and so FSP property follows for all solutions.

If $\min\{m, q^*, k^*\} \geq p$, then we have

$$m > p\beta, \quad q^*\beta > p\beta - 1 + \beta, \quad \text{and} \quad k^* > p\beta - 1 + \beta.$$

Thus for any $\omega > 0$ there exists $\sigma > 0$ so that

$$m\omega a z^{m-1+\beta} - a^{p+1} p\beta z^{p\beta-1+\beta} + \mu a^q z^{q\beta} + \lambda z^k \leq 0, \quad 0 < z < \sigma. \tag{3.8}$$

This means that $v(x,t) = z(\omega t - x + x_0)$ is a sub-solution. Since $\omega > 0$ is arbitrary, we may derive ISP property as in the proof of Lemma 2.7.

In the case (2) $\mu < 0$. Let $\min\{m, k^*\} < \min\{p, q\}$. As in the case (1) we may choose $\beta \in (0,1)$ close to 1 so that either

$$m \leq p\beta, \quad m - 1 + \beta \leq q\beta,$$

or

$$k^* \leq p\beta - 1 + \beta, \quad k^* \leq q\beta,$$

and then (3.7) holds for some $\omega \geq 0$ and $a > 0$, and consequently FSP follows.

The case $\min\{m, k^*\} \geq \min\{p, q\}$ can be divided into the following situations:

(a) $\min\{m, k^*\} > \min\{p, q\}$,
(b) $\min\{m, k^*\} = p \leq q$,
(c) $m > k^* = \min\{p, q\}$, and
(d) $k^* \geq m = q < p$.

In the first three cases we have that either

$$m - 1 + \beta > p\beta - 1 + \beta, \quad k^* \geq p\beta - 1 + \beta,$$

or

$$m - 1 + \beta > q\beta, \quad k^* \geq q\beta, \tag{3.9}$$

and hence (3.8) is valid for any $\omega > 0$ with some $\sigma > 0$.

The last situation (d) may be further divided into:

(d-1) $k^* \geq m = q < p$ and $m = q > 1$ — (3.9), and then (3.8), holds;
(d-2) $k^* \geq m = q < p$ and $m = q \leq 1$ — we have

$$m - 1 + \beta \leq q\beta, \quad m - 1 + \beta \leq p\beta - 1 + \beta,$$

and consequently (3.7), as well as FSP, is valid. □

4. **Multi-Dimensional Equation**

Now we consider n-dimensional equation (1.2), i.e.,

$$\partial_t \varphi(u) - \Delta_{p+1} u = 0 \ \text{ in } \ \mathbb{R}^n \times (0, \infty), \tag{4.1}$$

where $\Delta_{p+1} u = \operatorname{div}(|\nabla u|^{p-1} \nabla u)$. In a way parallel to one-dimensional case, super- and sub-solutions are defined similarly, and corresponding comparison principle keeps valid.

Theorem 4.1. *Let $u(x,t)$ be a nontrivial solution of* (4.1) *in* $\mathbb{R}^n \times (0,T)$. *If $\varphi(z)$ satisfies*

$$\lim_{z\to 0^+} \frac{\varphi'(z)}{|\log z|^p z^{p-1}} = 0,$$

then there exists $\tau \in (0,T]$ so that $\operatorname{supp} u(\cdot,t) = \mathbb{R}^n$ *for $0 < t < \tau$, i.e., $u(x,t) > 0$ for all $x \in \mathbb{R}^n$.*

Remark 4.2. This result shows that support of the solution, initially compact, will expand instantaneously out the whole space, namely, the perturbation propagates at infinite speed.

Proof. Without loss of generality, assume that $\operatorname{supp} u(\cdot,t) \supset\supset B_R$, the ball of radius $R > 0$ and centered at origin. By the continuity there exists $\tau \in (0,T]$ so that

$$\eta = \inf_{0<t<\tau, |x|\le R} u(x,t) > 0$$

since $u(x,0) > 0$ in $\overline{B}_R$. Now it suffices to show $u > 0$ in $Q := (\mathbb{R}^n \backslash B_R) \times (0,\tau)$.

Fix an arbitrary point $(x_0,t_0) \in Q$, set $\omega = (2|x_0| - R)/t_0$, and define $v(x,t) = z(\omega t - r + R)$, where $r = |x|$, $z(\zeta)$ solves the following equation:

$$\int_0^{z(\zeta)} \frac{ds}{f_\varepsilon(s)} = \zeta_+ \quad \text{for} \quad \zeta \le \zeta_0 := \int_0^\eta \frac{ds}{f_\varepsilon(s)},$$

where $f_\varepsilon(z) = [(z|\log z|)^{p-1} + \varepsilon z^{(p-1)\beta}]^{1/(p-1)}$, with $\beta \in (0,1)$ (say $\beta = 1/2$), and $\varepsilon > 0$ will be determined later. This gives

$$z' = f_\varepsilon(z), \quad 0 \le z(\zeta) \le \eta \quad \text{for} \quad \zeta \le \zeta_0.$$

By further calculation we obtain $\gamma, \sigma > 0$, not depending on ε, such that

$$z'' = f_\varepsilon'(z) z' \ge \gamma |\log z| z'$$

holds for $0 < z < \sigma$. Thus we have

$$\begin{aligned}
\partial_t \varphi(v) - \Delta_{p+1} v &= \omega \varphi'(z) z' + \partial_r \left[(z')^p\right] + \frac{n-1}{r}(z')^p \\
&= \omega \varphi'(z) z' - p(z')^{p-1} z'' + \frac{n-1}{r}(z')^{p-1} z' \\
&\le \left\{ \omega \varphi'(z) - (p\gamma |\log z| - \frac{n-1}{R})[(z|\log z|)^{p-1} + \varepsilon z^{(p-1)\beta}] \right\} z' \\
&\le \left\{ \omega \varphi'(z) - \frac{p\gamma}{2} |\log z|^p z^{p-1} \right\} z' \le 0
\end{aligned}$$

for $z \in (0,\sigma)$ with σ small enough. Now we may choose $\varepsilon > 0$ so small that

$$\zeta_0 = \int_0^\eta \frac{ds}{f_\varepsilon(s)} \ge \int_0^{\eta_0} \frac{ds}{f_\varepsilon(s)} \ge \omega\tau, \quad \eta_0 = \min\{\eta, \sigma\},$$

which implies that $v(x,t) \le z(\omega\tau) \le \eta_0$ when $(x,t) \in Q$. So $v(x,t)$ is a sub-solution of (4.1) in Q. In addition,

$$v(x,0) = 0 \le u(x,0) \quad \text{for} \quad |x| \ge R,$$

$$v(x,t) = z(\omega t) \le \eta \le u(x,t) \ \text{ for } \ |x| = R, \ 0 \le t \le \tau.$$

Therefore, comparison principle yields $v \le u$ in $Q = (\mathbb{R}^n \setminus B_R) \times (0,\tau)$. In particular,

$$u(x_0,t_0) \ge v(x_0,t_0) = z(\omega t_0 + R - |x_0|) > 0,$$

since $\omega t_0 + R - |x_0| = |x_0| > 0$. From the arbitrariness of (x_0,t_0) in Q it follows that

$$u(x,t) > 0 \ \text{ for all } |x| \ge R, \ t \in (0,\tau),$$

which completes the proof. □

Particularly, we may apply above conclusion to the equation

$$\partial_t(u^m) - \Delta_{p+1} u = 0 \ \text{ in } \ \mathbb{R}^n \times (0,T). \tag{4.2}$$

Corollary 4.3. *Equation* (4.2) *admits infinite speed propagation if $m \ge p$.*

The following result implies that (4.2) admits FSP property if $m < p$.

Theorem 4.4. *Let $u(x,t)$ be a bounded solution of* (4.1) *in $\mathbb{R}^n \times (0,T)$, satisfying*

$$\operatorname{supp} u(\cdot,0) \subset \Omega_0 = \{x = (x_1,x') \in \mathbb{R}^n |\ x_1 < 0\}.$$

If $\int_0^\sigma \frac{dz}{\varphi^{1/p}(z)} < \infty$, then there exist $\tau \in (0,T]$ and $r(t) > 0$ so that

$$\operatorname{supp} u(\cdot,t) \subset \Omega_{r(t)} = \{x = (x_1,x') \in \mathbb{R}^n |\ x_1 < r(t)\}, \quad 0 < t < \tau.$$

In other words,

$$u(x,t) = 0 \ \textit{ for } \ x_1 \ge r(t), \quad 0 < t < \tau.$$

Proof. In order to show FSP property of u we construct a super-solution as follows.

Let $\omega > 0$, $z(\zeta)$ solves the following equation:

$$\int_0^{z(\zeta)} \frac{ds}{[\omega\varphi(s)]^{1/p}} = \zeta_+ < +\infty, \text{and} \quad \zeta_0 := \int_0^M \frac{ds}{[\omega\varphi(s)]^{1/p}}.$$

where $M = \max u(x)$. Then $z'(\zeta) = [\omega\varphi(z)]^{1/p}$, and $z(\zeta) \ge 0$.

Now consider function $v(x,t) = z(\omega t - x_1 + \zeta_0)$ where $x = (x_1,x')$, $x' \in \mathbb{R}^{n-1}$ in $P_T = \mathbb{R}^n_+ \times (0,T)$, $\mathbb{R}^n_+ = \{x = (x_1,x')|x_1 \ge 0, x' \in \mathbb{R}^{n-1}\}$. Then we have $v(x,t) \ge 0$, and

$$\begin{aligned}\partial_t\varphi(v) - \Delta_{p+1} v =& \varphi'(z)\omega z' + \partial_{x_1}(z')^p \\ =& \omega\varphi'(z)z' - \omega\varphi'(z)z' = 0.\end{aligned}$$

Thus $v(x,t)$ is a super-solution of (4.1) in P_T. Besides, we see that

$$u(x,0) = 0 \le v(x,0) = z(-x_1 + x_0) \ \text{ for } x_1 \ge 0;$$

and by the boundedness of solution so that

$$u(0,x',t) \le M = z(\zeta_0) \le z(\omega t + \zeta_0) = v(0,x',t) \ \text{ for } \ 0 \le t \le T.$$

According to the comparison principle we obtain that $u \le v$ on $\overline{P}_T = \mathbb{R}^n_+ \times [0, T]$. Particularly,

$$u(x,t) = 0 \quad \text{for} \quad x_1 \ge \omega t + \zeta_0, \quad 0 \le t \le \tau,$$

i.e., FSP is admitted. □

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